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ANALYTICAL AND NUMERICAL RESEARCH OF THE FORMS OF EQUILIBRIUM OF A LIQUID LIMITED VOLUME ON A ROTATING HORIZONTAL BASE

The article studies axisymmetric forms of drop relative equilibrium on a rotating disk due to the action of gravitational, centrifugal and capillary forces. It is obtained that there are two main types of forms of equilibrium: with simply connected and not simply connected sets of points of contact of liquid and solid phases. A number of statements about these forms of equilibrium are proved analytically. The findings are consistent with the numerically found drop shapes for various examples of the input data. Curves for determination of dependencies between various parameters of liquid volume in relative rest are constructed.

Keywords: *rotating disk, relative equilibrium, Navier—Stokes equations, Laplace formula, surface tension, contact angle, axisymmetric drop*

Introduction. In various industries, processes that use the movement of a liquid layer on the surface of a rotating disk are widely used. These include the phenomena of fiber formation by centrifugal-disk method, application of coating and lubrication on a flat surface by rotation. The problems of determining the shape of the free surface of a rotating limited volume of liquid and the movement of a liquid layer on a rotating solid body belong to fundamental problems in the field of hydromechanics. The book [1] is an example of studying the figures of relative equilibrium of a liquid in a rotating coordinate system, taking into account the external pressure and self-gravity of the liquid. The work [2] is devoted to the mathematical study of the behavior of a liquid in conditions of full or partial weightlessness, implemented during space flights. The book [3] addresses the issue of flow around a rotating disk in both free and confined space, and also examines the flow around some other bodies, there is a comparison with experimental data. The paper [4] considers the problem of flat stationary motion of a thin layer of a viscous incompressible liquid on the surface of a circular horizontal cylinder rotating around its axis in the field of gravity. In the articles [5, 6], the study also covers the non-stationary motion of the liquid layer. In the framework of the paper [7], an upper bound estimation is found for the stability margin of a circular cylindrical equilibrium state of a rotating liquid enclosed between two parallel plates. In the work [8], the stability of stationary axisymmetric flows to non-axisymmetric perturbations is studied. The articles [9, 10] are devoted to the study of the forms of relative equilibrium of axisymmetric and non-axisymmetric droplets on a rotating horizontal base and the comparison of theoretical results with experimental data. In the work [11], the results of numerical, analytical and experimental studies of a viscous liquid layer on a rotating vertical disk are presented. Experimental studies have shown that at low rotational speeds, the relative equilibrium of the layer on the disk is possible [10]. As the rotation speed increases, the layer splits into trickles in the radial direction. In this paper, special attention is paid to analytical studies of the relative rest of isolated droplet layers and layers from the annular contact area of the liquid and solid phases.

Problem statement. The behavior of a liquid drop is considered which is located on a horizontal base rotating around an axis perpendicular to its surface with a constant angular velocity. It is assumed that the drop has a rotational symmetry relative to the axis and is in a state of equilibrium in a mobile coordinate system that is rigidly connected to the substrate. The contact angle α, the modulus of the gravitational acceleration g, the drop mass M, the fluid density ρ, the surface tension coefficient σ, atmospheric pressure p_a, and the angular velocity modulus ω are known. We introduce a relative cylindrical coordinate system (r, φ, z), z axis of which coincides with the axis of disk rotation and is directed upwards. The field of forces of the mass unit in this system is decomposed into the centrifugal r-component $F_r = \omega^2 r$, and the z-component $F_z = -g$. Taking this into account, as well as the relative rest of the liquid, the Navier–Stokes equations in the cylindrical coordinate system can be integrated to find the pressure distribution p in the drop:

$$p = \frac{1}{2}\rho\omega^2 r^2 - \rho g z + p_1, \tag{1}$$

where p_1 is a constant that has the meaning of the liquid pressure at the point $r = 0$, $z = 0$, if there is a liquid at this point. Let the curve Λ be the result of the intersection of the free surface of a drop with a certain half-plane φ = const. Due to the symmetry of the drop, this curve completely determines the shape of its free surface in the system (r, z). We choose the positive direction of circumventing the curve Λ so that when going around the area occupied by the liquid is located on the right, provided that to align the r axis with the z axis, the first one must be rotated counterclockwise at an angle of 90°. Let θ be the counterclockwise measured angle between the positive direction of the r axis and the tangent vector to the curve under consideration at this point, indicating the positive direction of circumvent Λ. Let us set:

$$x = \sin\theta,\ y = -\cos\theta. \tag{2}$$

We introduce dimensionless quantities r_0, z_0, Δp_0 by relations $r = Rr_0$, $z = Rz_0$, $p_1 - p_a = \rho\omega^2 R^2 \Delta p_0$, and Weber and Froude numbers by formulas $We = \rho\omega^2 R^3/\sigma$, $Fr = \omega^2 R/g$. Then, taking into account (1), the Laplace formula [2] for surface tension can be written as

$$\begin{gathered} \frac{x}{r_0} + \frac{dy}{dz_0} = \frac{x}{r_0} + \frac{dx}{dr_0} = \\ = We\left(-\frac{1}{2}r_0^2 + Fr^{-1}z_0 - \Delta p_0\right) = f\left(r_0, z_0\right). \end{gathered} \tag{3}$$

Equations (3) are convenient for determining a number of qualitative properties of possible curves Λ.

If the surface of the drop contact with the disk is a circle, then the task of determining the drop shape will be called a task of type I; a task of type II is the task corresponding to the annular contact region of the liquid and solid phases. We introduce a natural parametrization $r_0 = r_0(t)$, $z_0 = z_0(t)$ of the curve Λ, where t is a natural parameter. Let the parameter t increase with a positive circumvent of the curve Λ, and let the beginning of the curve correspond to the value $t = 0$, and its end — $t = T_{\text{end}}$. Then from (2), (3) we get the system:

$$\begin{aligned} &\frac{d^2 z_0}{dt^2} r_0 - f\left(r_0, z_0\right) r_0 \frac{dr_0}{dt} + \frac{dr_0}{dt}\frac{dz_0}{dt} = 0, \\ &\frac{d^2 r_0}{dt^2} r_0 + f\left(r_0, z_0\right) r_0 \frac{dz_0}{dt} - \left(\frac{dz_0}{dt}\right)^2 = 0. \end{aligned} \tag{4}$$

The system (4) is convenient for numerical determination and investigation of the forms of relative equilibrium of the drop. It must be supplemented with boundary conditions and an integral condition of a given mass. For the task of type I, they look like:

$$\begin{aligned} &\frac{dz_0}{dt}(0) = 0;\ \frac{dr_0}{dt}(0) = 1;\ \frac{dz_0}{dt}\left(T_{end}\right) = -\sin\alpha; \\ &\frac{dr_0}{dt}\left(T_{end}\right) = \cos\alpha;\ r_0(0) = 0,\ z_0\left(T_{end}\right) = Z_0. \end{aligned} \tag{5}$$

For the task of type II, the boundary conditions have the following form:

$$\begin{gathered} \frac{dz_0}{dt}(0) = -\frac{dz_0}{dt}\left(T_{end}\right) = \sin\alpha;\ \frac{dr_0}{dt}(0) = \frac{dr_0}{dt}\left(T_{end}\right) = \cos\alpha; \\ z_0(0) = z_0\left(T_{end}\right) = Z_0. \end{gathered} \tag{6}$$

The drop mass is set by the ratio

$$M = 2\pi\rho R^3\left(\int_0^{T_{end}} r_0(t) z_0(t)\frac{dr_0}{dt}(t)dt - \frac{Z_0}{2}\left(r_0\left(T_{end}\right)^2 - r_0(0)^2\right)\right). \tag{7}$$

Here Z_0 is the z_0-coordinate of any point on the disk surface. Of the four boundary conditions imposed on the derivatives of functions $z_0(t)$ and $r_0(t)$, it is sufficient to select three and take into account the sign on the right side of the fourth condition.

Results of the analytical study of forms of equilibrium. Analysis of system (4) makes it possible to draw the following conclusions:

1. There are no two different tangent curves that satisfy system (4) and have the same curvature vector at the tangent point.
2. For a bounded drop, there are no straight sections of the Λ curve.
3. There are no inflection points of the Λ curve where the tangent to this curve is parallel to the axis r_0 or axis z_0.

Analysis of the system (2), (3) leads to the following statements:

1. The parameterization function $z_0(t)$ cannot have more than one local maximum, or, equivalently taking into account its continuity, cannot have local minima on the interval $(0, T_{\text{end}})$.
2. The parameterization function $r_0(t)$ cannot have a local maximum on the interval $(0, T_{\text{end}})$, followed by a local minimum belonging to this interval with increasing argument t.

These statements allow us to draw the following conclusions about possible forms of equilibrium of the drop on a rotating disk. By investigating the task of type I, it is possible to obtain convex drops if the curve Λ has only a decreasing section of the function $z_0(t)$. If the curve Λ has an increasing section of both functions $r_0(t)$ and $z_0(t)$, which is replaced by a decreasing section of the function $z_0(t)$ when the curve is positively traversed, we get drops with a concavity zone, as is seen in Figure 1 *a*. In this case, if the contact angle is acute or right, then $r_0(t)$ grows over the entire curve. If the contact angle is obtuse, then the decreasing section $z_0(t)$ consists of the increasing section $r_0(t)$, which is replaced by the decreasing section of the function $r_0(t)$ as t increases.

In the task of type II, with a ring contact area at an obtuse contact angle, the section of the decreasing function $r_0(t)$ and increasing function $z_0(t)$ with a positive circumvent of the curve Λ is replaced by the increasing section of both functions. Then there is a section of decreasing function $z_0(t)$ and increasing function $r_0(t)$, which in turn is followed by a section of decreasing both functions, as in Figure 1 *b*. In this case, we set with P_1 and P_2 the planes parallel to the surface of the disk and passing through the nearest $R_{\min}$ to the z axis and farthest from it points $R_{\max}$ of the curve Λ, respectively. If the contact angle is acute or right, then the first and last named sections are missing. Let the area where the function $r_0(t)$ increases be indicated as Λ_2. This section is present in the tasks of both types for all forms of the drop, and it is the only one. If the contact angle is greater than 90°, then there is also a section Λ_3, where $r_0(t)$ and $z_0(t)$ are decreasing functions, and for the task of type II there will also be a section Λ_1, where $r_0(t)$ decreases and $z_0(t)$ increases. In the task of type II and in the case of a drop with a concavity zone in the task of type I, the section Λ_2 consists of sections Λ_{2a} and Λ_{2b}, where $z_0(t)$ increases and decreases respectively. For the task of type I, we define the P_2 plane in the same way as for the task of type II.

The study of the system (2), (3) also makes it possible to make the following observations about the possible forms of the curve Λ:

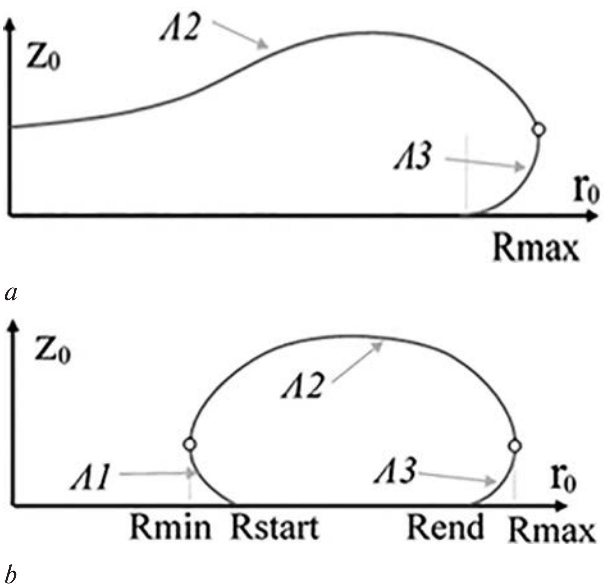


Figure 1 — Schematic representation of layers of:
a — type I; *b* — type II

1. There are no inflection points on sections Λ_{2b} and Λ_3. Moreover, the curvature vector at any point of these sections is directed inside the drop.

2. In the task of type I, if a drop has a concavity zone, there is only one inflection point of the curve Λ.

3. Let us consider the task of type II with a contact angle greater than 90°, and the point A of the section Λ_1 and the point B of the section Λ_2 are equally spaced from the z axis. Then the distance from point A to P_1 and the value of x at point A do not exceed, respectively, the distance from point B to P_1 and the value of x at point B. If the curvature vector at point B is directed inside the drop, then at point A the curvature vector is also directed inside, and its modulus is greater.

4. Let the contact angle be greater than 90°, and point A of the section Λ_3 and point B of the section Λ_2 are equally spaced from the z axis. Then the distance from point A to P_2, the value of x at point B, and the modulus of the curvature vector at point B do not exceed, respectively, the distance from point B to P_1 and the value of x at point A and the value of the curvature vector at point A.

5. The highest point of the drop is always no closer to the axis z_0 than the beginning of the curve Λ, and no further from this axis than the end of this curve.

Analysis of the system (2), (3) taking into account the integral and boundary conditions (5), (7) leads to the following conclusions:

1. For the task of type I with settled ρ, σ, $\omega > 0$, $g > 0$, α, increasing the drop radius will first lead to the impossibility of the existence of a convex drop, then to the impossibility of the existence of a solution of the system (4), (5), (7), complying with the physical and mathematical sense.

2. With the settled ρ, σ, $\omega > 0$, $g > 0$, α, there is such a value of mass M_1 that a drop of mass $M > M_1$ cannot exist.

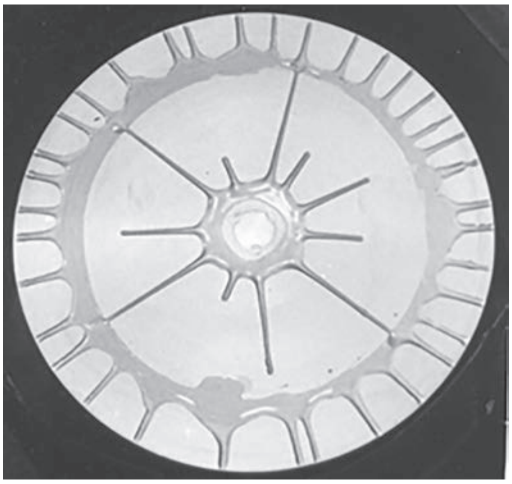
Figure 2 shows the results of experimental studies of the movement and decay of isolated ring-shaped layers of the oil PFMS-4. With slow rotation, as is seen in Figure 2 *a*, it is possible to rotate the disk and the layer as a whole. With an increase in the angular speed of rotation, the stream-threads fly in the radial direction.

Results of numerical study of forms of equilibrium. In the task of type I, all parameters of the drop at a settled characteristic distance are uniquely determined by the value $x'(0)$, where the stroke means the derivative with respect to r_0. The modulus of this value is equal to the curvature of the curve Λ, given in dimensionless coordinates (r_0, z_0), at a point lying on the axis z_0 or the curvature of the drop in the center. Let R_F be the radius of the drop with $x'(0) = 0$, and X_F be the value of $x'(0)$ with the characteristic distance R_F.

The Table shows the values of physical quantities for three sets of parameters used in numerical calculations.

From Figure 3, we can conclude that the drop radius increases with the curvature in the center. The same result can be reached analytically in the course of studying the system (2), (3).

a



b

Figure 2 — Shape of the ring-shaped layers of the oil PFMS-4 on the disk: *a* — without rotation; *b* — with rotation

Figures 6 and 7 show graphs that reflect the possible shapes of the Λ curve under various conditions for the task of type I. The graphs are given in dimensionless coordinates, where the characteristic distance is the radius of the drop represented by a solid line in the corresponding figure. In all cases, $p = 1000$ kg/m^3, $\sigma = 0.07286$ N/m. In the Figures 6 and 7, the solid line corresponds to a weightless drop with values $g = 0$ m/s^2, $\omega = 0$ rad/s, the dashed line with short strokes — $g = 9.81$ m/s^2, $\omega = 8\pi$ rad/s, the dashed line with long strokes — $g = 15$ m/s^2, $\omega = 4\pi$ rad/s, the dash-dotted line — $g = 4$ m/s^2, $\omega = 4\pi$ rad/s, the dotted line — $g = 1$ m/s^2, $\omega = 8\pi$ rad/s. The contact angle in the Figure 6 is equal to $\alpha = 60°$, the mass of the drop in this Figure $M = 0.0004$ kg. In the Figure 7: $\alpha = 120°$,

Table — Parameters corresponding to various graphs in the Figures 3–5

	Set of parameters I	Set of parameters II	Set of parameters III
Line	Solid line	Dotted line	Dashed line
ρ, kg/m^3	1000	1000	800
σ, N/m	0.07286	0.07286	0.04
α	60°	120°	90°
ω, rad/s	6π	7π	8π
g, m/s^2	9.81	9.81	9.81

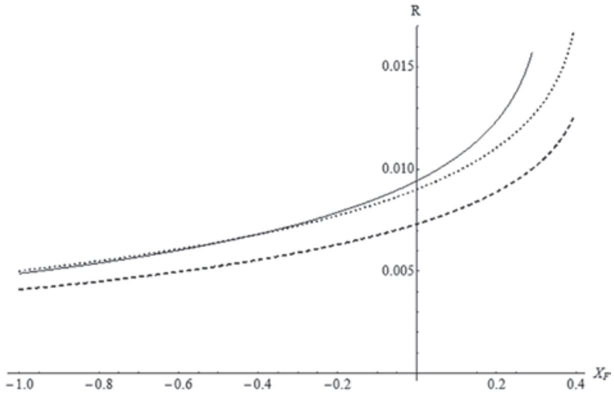


Figure 3 — Dependence of the radius *R* of the drop on the curvature in its center

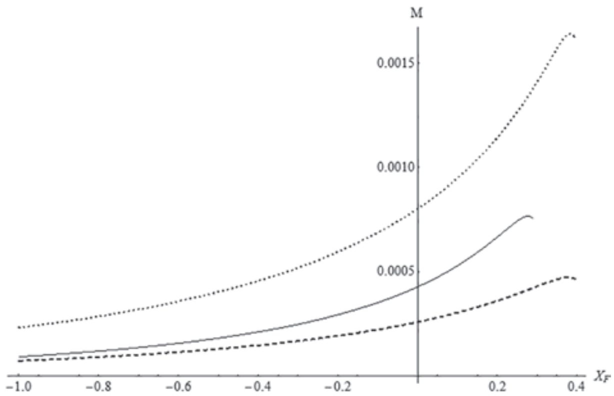


Figure 4 — Dependence of the mass *M* of the drop on the curvature in its center

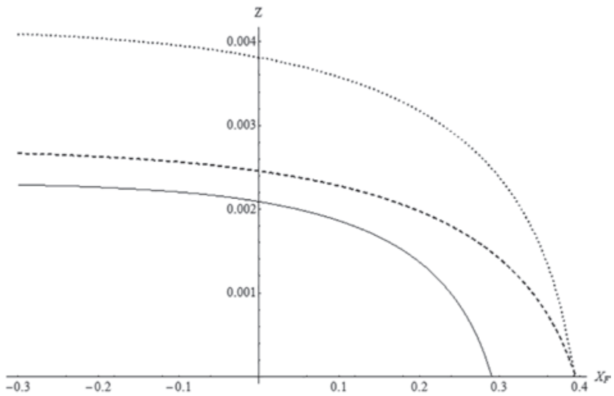


Figure 5 — Dependence of the height *Z* of the drop in the middle on the curvature in the center

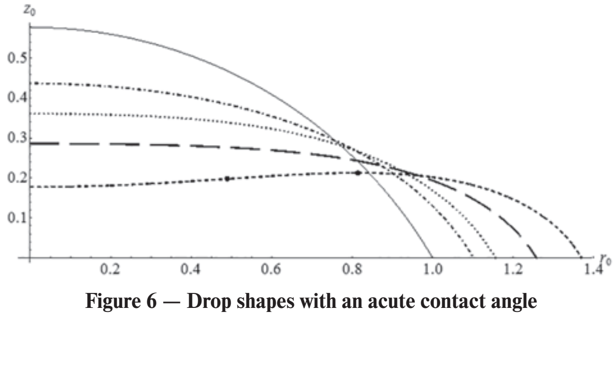


Figure 6 — Drop shapes with an acute contact angle

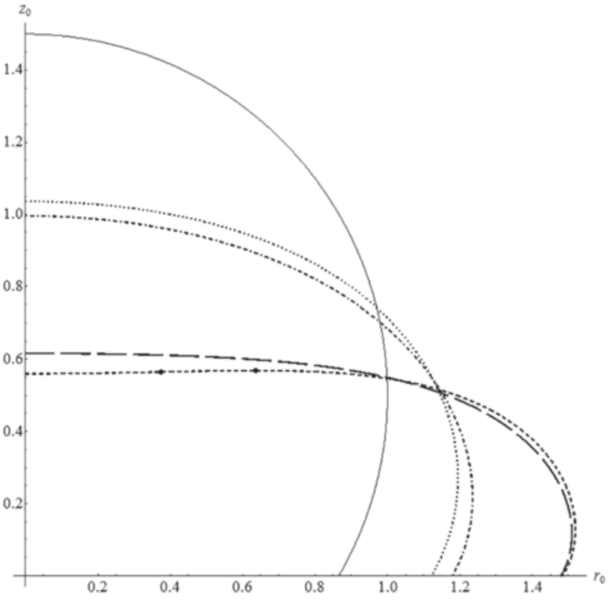


Figure 7 — Drop shapes with an obtuse contact angle

$M = 0.0008$ kg. The inflection point in the section Λ_{2a}, if there is, and the endpoint of this section are marked with points. It can be concluded that an increase in gravity contributes to greater pressure of the drop to the disk. On the graphs, the disk surface is selected as the zero level z_0, i. e. $Z_0 = 0$. When constructing the curves, we used the fact that the shape of the curve Λ is uniquely dependent on the curvature at its beginning and the method of targeting. It is based on the secant method to select curvature in the center of the drop, which is necessary for the drop mass to correspond to the specified value of M with the required accuracy.

Conclusion. In this paper, the equations of relative rest of the liquid drop and the isolated symmetric volume of a liquid layer on a rotating disk are obtained, analytical studies and numerical solutions of these equations are carried out depending on the characteristic parameters of the problem. The forms of equilibrium of the liquid limited volume obtained by numerical methods, as well as the qualitative dependences between some parameters of the drop, are consistent with the results of analytical study of the mathematical model and experimental results [10]. The research results can be used to analyze the thickness of the coating and lubricant on the rotating horizontal surface of the disk.

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АНАЛИТИЧЕСКОЕ И ЧИСЛЕННОЕ ИССЛЕДОВАНИЕ ФОРМ РАВНОВЕСИЯ ОГРАНИЧЕННОГО ОБЪЕМА ЖИДКОСТИ НА ВРАЩАЮЩЕМСЯ ГОРИЗОНТАЛЬНОМ ОСНОВАНИИ

Исследуются осесимметричные формы относительного равновесия капли на вращающемся диске, обусловленные действием гравитационных, центробежных и капиллярных сил. Получено, что возможны два основных типа форм равновесия: с односвязным и неодносвязным множеством точек контакта жидкой и твердой фаз. Аналитически доказан ряд утверждений о данных формах равновесия. Полученные выводы согласуются с численно найденными формами капель и экспериментальными результатами. Построены кривые для определения зависимостей между различными параметрами объема жидкости в ее относительном покое.

Ключевые слова: *вращающийся диск, относительное равновесие, уравнения Навье–Стокса, формула Лапласа, поверхностное натяжение, краевой угол смачивания, осесимметричная капля*

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